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Problems and conjectures concerning connectivity, paths, trees and cycles in tournament-like digraphs

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ABSTRACT

In this paper we collect a substantial number of challenging open problems and conjectures on connectivity, paths, trees and cycles in tournaments and classes of digraphs which contain tournaments as a subclass. The list is by no means exhaustive but is meant to show that the area has a large number of interesting open problems. We also mention problems for general digraphs when they are relevant in the context.

1. Introduction

Tournaments are no doubt the most well-studied class of directed graphs and within the last 15 years a lot of attention has been given to classes of digraphs which contain tournaments as a subclass. These include locally semicomplete digraphs, quasi-transitive digraphs, in-tournaments, path-mergeable digraphs and in particular multipartite tournaments. For more literature on these classes as well as for general notation of this paper, the reader is referred to [8]. Even though these classes have very nice structure which allows us to solve many problems such as the hamiltonian cycle problem efficiently, there are still many challenging problems. Already for digraphs which are very close to tournaments, say a tournament with just a few arcs deleted, problems such as the hamiltonian cycle problem become very complicated.

Besides being of interest to people working on digraphs some of these problems and conjectures may turn out to be relevant also to a broader audience. As an example let us consider the conjecture made by the author and Thomassen in 1992 [17] that the feedback arc set problem, that is, finding a minimum set of arcs that meet all cycles, is NP-hard for tournaments. Within the last couple of years at least four different groups have worked on this conjecture: Ailon, Charikar and Newman [2] first proved that the problem is NP-hard under randomized reductions. Then Alon [3] showed how to reduce the feedback arc set problem for general digraphs to the same problem for tournaments in polynomial time, thereby settling the conjecture. Slightly later the same result was obtained independently by Charbit, Thomassé and Yeo [22]. Finally Conitzer [25] gave a different proof by describing a deterministic reduction of MAXSAT to the feedback arc set problem for tournaments.

Problem 1 (*Ailon and Alon [1]*). Is there a polynomial time approximation scheme for the feedback arc set problem for the class of tournaments.

It was shown in [48] that the feedback arc set problem is fixed parameter tractable for tournaments.

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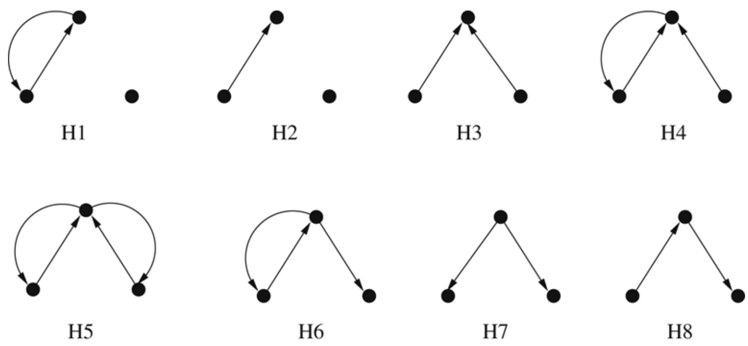

Fig. 1. Eight digraphs on three vertices. The arcs shown are the only ones present.

2. Basic terminology

We refer to [8] for general definitions on (di)graphs and for a large number of results on various classes of digraphs that contain the tournaments as a subclass.

A digraph $D = (V, A)$ is

- **semicomplete** if it has no non-adjacent vertices.
- **in-semicomplete** (**out-semicomplete**) if it does not contain any of H_3, H_4, H_5 (H_5, H_6, H_7) as an induced subdigraph (see Fig. 1).
- **locally semicomplete** if it is both in-semicomplete and out-semicomplete.
- **semicomplete multipartite** if it contains none of H_1, H_2 as an induced subdigraph. If it also has no 2-cycles it is a **multipartite tournament**.
- **quasi-transitive** if it contains none of H_4, H_5, H_6, H_8 as an induced subdigraph.
- **path-mergeable** if for every pair of vertices x, y and every pair of distinct (x, y)-paths P_1, P_2 there exists another (x, y)-path P such that $V(P) = V(P_1) \cup V(P_2)$.
- **extended semicomplete** if it can be obtained from a semicomplete digraph S by substituting independent sets for the vertices, i.e. D has the form $D = S[I_1, I_2, \ldots, I_s]$, where S is semicomplete and each I_j is an independent set of vertices. Hence if s_i and s_j are distinct vertices of S and $s_i \rightarrow s_j$ is an arc of S, then $x \rightarrow y$ is an arc of D for every $x \in I_i$ and $y \in I_j$.
- k-**arc-strong** if $D - X$ is strongly connected for every subset $X \subset A$ with $|X| < k$. We denote by $\lambda(D)$ the arc-strong connectivity of D, that is, the maximum k for which D is k-arc-strong.
- k-**strong** if D has at least $k + 1$ vertices and $D - X$ is strongly connected for every subset $X \subset V$ with $|X| < k$. For $k = 1$ we also consider the digraph on just one vertex as a 1-strong digraph.

For a digraph D we denote by $UG(D)$ the underlying undirected graph of D, that is, the graph that we obtain by suppressing all orientations and deleting multiple edges.

We denote by $\alpha(D)$ the *independence number* of D which is the size of a largest independent set of vertices in $UG(D)$.

The minimum in-degree (out-degree) of a digraph is denoted by $\delta^-(D)$ ($\delta^+(D)$). The **minimum degree** of D, $\delta(D)$, is the minimum of $\delta^-(D)$ and $\delta^+(D)$. The in-degree (out-degree) of a subset $X \subset V(D)$ denoted by $d^-(X)$ ($d^+(X)$) is the number of arcs going from $V(D) - X$ to X (X to $V(D) - X$).

A *cycle factor* in a digraph D is a collection of disjoint cycles which contain all vertices in D. The existence of a cycle factor can be checked in polynomial time using flows (see [8, Section 3.11]).

3. Longest paths and cycles

Bang-Jensen, Gutin and Yeo proved in [11] that the hamiltonian cycle problem can be solved in polynomial time for semicomplete multipartite digraphs. However, neither the algorithm described there nor the faster $O(n^5)$ algorithm by Yeo in [58] can be used to find a longest cycle in a semicomplete multipartite digraph.

Problem 2 (*Bang-Jensen, Gutin and Yeo [11]*). Is there a polynomial algorithm for finding a longest cycle in a semicomplete multipartite digraph?

Path-mergeable digraphs are hamiltonian whenever the two obvious conditions of being strongly connected and having no cutvertex in the underlying graph are satisfied [6]. The proof of this fact is a generalization of the standard proof of Camion's theorem that every strong tournament is hamiltonian and leads to a polynomial algorithm. The hamiltonian path problem, however, seems much harder and it is not clear how to use the path-merging property to construct a hamiltonian path.

Problem 3 (*Bang-Jensen [6]*). What is the complexity of the hamiltonian path problem for path-mergeable digraphs?

A digraph D is *hamiltonian connected* if it contains an (x, y)-hamiltonian path for every choice of distinct vertices $x, y \in V(D)$. Thomassen [51] proved that every 4-strong semicomplete digraph is hamiltonian connected. In [15] Bang-Jensen, Manoussakis and Thomassen used this result and several other structural results on hamiltonian paths in semicomplete digraphs to develop a polynomial algorithm for testing whether a given semicomplete digraph contains an (x, y)-hamiltonian path. This algorithm is based on a divide-and-conquer approach where the current problem may be reduced to up to 4 different smaller problems of the same type. Interestingly, the approach used in [15] does not seem easily adapted to the problem of finding the longest (x, y)-path in a semicomplete digraph, nor does it seem that this problem can be reduced to the (x, y)-hamiltonian path problem.

Problem 4 (*Bang-Jensen and Gutin [7]*). Is there a polynomial algorithm for finding a longest (x, y)-path in a semicomplete digraph? Is there a structural characterization?

Volkmann suggested the following generalization of Thomassen's theorem to semicomplete multipartite digraphs.

Conjecture 1 (*Volkmann [56]*). *Every $(\alpha(D) + 3)$-strong semicomplete multipartite digraph is hamiltonian connected.*

Clearly, if a digraph D has a hamiltonian (x, y)-path then it has an (x, y)-path P so that $D - V(P)$ can be covered by vertex-disjoint cycles (the empty collection). The following conjecture suggests that for a subclass of semicomplete multipartite digraphs being 4-strong, this condition suffices to guarantee a hamiltonian (x, y)-path.

Conjecture 2 (*Bang-Jensen, Gutin and Huang [9]*). *Let D be a 4-strong extended semicomplete digraph with an (x, y)-path P such that $D - V(P)$ has a cycle factor. Then D contains a hamiltonian (x, y)-path.*

4. Trees in tournaments

4.1. Sumner's conjecture

Conjecture 3 (*Sumner [57]*). *Every tournament on $2(n - 1)$ vertices contains every oriented tree with n vertices.*

The best bound shown so far is due to El Sahili who proved that every tournament on $3(n-1)$ vertices contain all oriented trees on n vertices [29].

4.2. Arc-disjoint branchings with few leaves

An *out-branching* (*in-branching*) rooted at s in a directed multigraph D is a spanning tree in $UG(D)$ which is oriented in such a way that every vertex except s has precisely one arc entering (leaving). See Fig. 2.

Edmonds proved [28] that a digraph D contains k arc-disjoint out-branchings rooted at the vertex s if and only if

$$d^-(X) \geq k \quad \text{for every } X \subseteq V - s \tag{1}$$

or, equivalently (by Menger's theorem), s has k arc-disjoint paths to every other vertex.

A *leaf* of an out-branching F_s^+ is a vertex of out-degree zero in F_s^+. By a slight modification of the Gallai–Millgram theorem and Edmond's branching theorem (see [8, Pages 234 and 501]), every 2-arc-strong tournament contains arc-disjoint out-branchings $F_{s,1}^+, F_{s,2}^+$ such that each has at most two leaves. Note that removing the edges of a tree from a complete graph results in a graph with independence number two.

Problem 5. When does a tournament contain two arc-disjoint out-branchings $F_{s,1}^+, F_{s,2}^+$ both rooted in the same vertex such that one of these is a hamiltonian path from s?

The example in Fig. 3 shows that 2-strong connectivity is not sufficient to guarantee arc-disjoint out-branchings $F_{s,1}^+, F_{s,2}^+$ such that one of these is a hamiltonian path from s. Note that if the roots do not have to be the same, then every 2-strong tournament has such branchings (see Section 7).

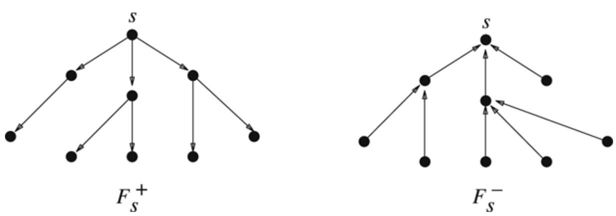

Fig. 2. An out-branching F_s^+ rooted at s and an in-branching F_s^- rooted at s.

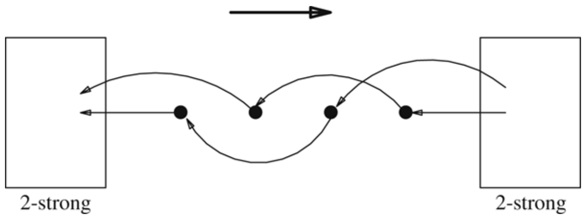

Fig. 3. A 2-strong tournament. Each of the two boxes represents 2-strong tournaments and all arcs except the six explicitly shown go from left to right.

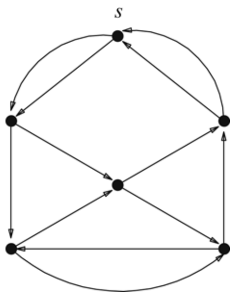

Fig. 4. A 2-regular 2-arc-strong directed multigraph with no arc-disjoint in- and out-branchings F_s^-, F_s^+ rooted at s.

4.3. Arc-disjoint in- and out-branchings

Thomassen (see [5]) proved that deciding whether a digraph D has arc-disjoint in- and out-branchings F_v^-, F_v^+ with the same root is NP-complete. Bang-Jensen proved [5] that every 2-arc-strong tournament T has arc-disjoint in- and out-branchings F_v^-, F_v^+ for every choice of $v \in V(T)$ and gave a polynomial algorithm for deciding whether a given tournament T has arc-disjoint in- and out-branchings F_u^-, F_v^+ for given $u, v \in V(T)$.

In fact a much stronger result holds: Bang-Jensen and Yeo [19] have shown that every $74k$-arc-strong tournament contains $2k$ arc-disjoint branchings $F_{u_1}^-, \ldots, F_{u_k}^-, F_{v_1}^+, \ldots, F_{v_k}^+$.

Conjecture 4 (*Bang-Jensen and Yeo [19]*). *Every $2k$-arc-strong tournament contains $2k$ arc-disjoint branchings $F_{u_1}^-, \ldots, F_{u_k}^-, F_{v_1}^+, \ldots, F_{v_k}^+$.*

Conjecture 5 (*Bang-Jensen and Gutin [7]*). *There is a polynomial algorithm for deciding whether a given digraph D which is either locally semicomplete or quasi-transitive has arc-disjoint in- and out-branchings F_u^-, F_v^+ for given $u, v \in V(D)$.*

Conjecture 6 (*Thomassen [53]*). *Every 10^{10}-arc-strong digraph D has arc-disjoint in- and out-branchings F_v^-, F_v^+ for every choice of $v \in V(D)$.*

Very little is known about this problem for general digraphs. Fig. 4 shows a 2-arc-strong directed multigraph with no arc-disjoint in- and out-branchings rooted at the vertex s.

Problem 6. Find a 3-arc-strong digraph D which does not have arc-disjoint in- and out-branchings F_v^-, F_v^+ for some choice of $v \in V(D)$.

Conjecture 6 would follow immediately from Conjecture 12 in the next section.

5. Decompositions

5.1. Decompositions into arc-disjoint spanning subdigraphs

The so-called Kelly conjecture[1] states that every regular tournament on $2k + 1$ vertices has a decomposition into k-arc-disjoint hamiltonian cycles. Note that it is an easy exercise to show that every k-regular tournament is k-arc-strong.

Conjecture 7 (*Kelly [46]*). *Every k-regular tournament contains k arc-disjoint hamiltonian cycles.*

[1] A proof of the Kelly conjecture for large k has been announced by R. Häggkvist at several conferences and in [21] but to this date no proof has been published.

In [19] a generalization of the Kelly conjecture is given.

Conjecture 8 (*Bang-Jensen and Yeo [19]*). *Every k-arc-strong tournament decomposes into k spanning strong digraphs.*

The paper [19] contains several results which support the conjecture

- If $D = (V, A)$ is a 2-arc-strong semicomplete digraph then it contains 2 arc-disjoint spanning strong subdigraphs except for one digraph on 4 vertices.
- The conjecture is true for every tournament (in fact every semicomplete digraph) which has a non-trivial cut (both sides of size at least 2) with precisely k arcs in one direction.
- Every k-arc-strong tournament with minimum in- and out-degree at least $37k$ contains k arc-disjoint spanning subdigraphs $H_1, H_2, \ldots, H_k$ such that each H_i is strongly connected.

Conjecture 8 implies the Kelly conjecture. On the other hand, one can construct k-arc-strong tournaments T on arbitrarily many vertices for which $\lambda(T - x) < k$ for every $x \in V(T)$ (for instance by modifying slightly the idea used by Thomassen on page 166 of [52]). Hence Conjecture 8 does not seem to follow easily from the Kelly Conjecture. The following two conjectures represent successive weakenings of Conjecture 8.

Conjecture 9 (*Bang-Jensen and Yeo [19]*). *Let k, s and t be natural numbers such that $k = s + t$. Then every k-arc-strong tournament contains arc-disjoint spanning strong subdigraphs D_1, D_2 such that D_1 is s-arc-strong and D_2 is t-arc-strong.*

Conjecture 10 (*Bang-Jensen and Yeo [19]*). *Every k-arc-strong tournament T contains a spanning strong subdigraph H such that $T - A(H)$ is $(k - 1)$-arc-strong.*

Thomassen proved [52, Theorem 4.2] that every 2-arc-strong tournament T contains a hamiltonian path P such that $T - A(P)$ is strong. It is interesting to note that we cannot replace the hamiltonian path by a hamiltonian cycle above, as shown by the infinite class of 2-arc-strong tournaments in Fig. 3.

Conjecture 11 (*Bang-Jensen and Yeo [19]*). *Except for finitely many exceptions for each k, every k-arc-strong semicomplete digraph can be decomposed into k arc-disjoint spanning strong subdigraphs.*

For $k = 2$ there is precisely one exception on 4 vertices [19].

For general digraphs almost nothing is known about decompositions into spanning strong subdigraphs and furthermore it is NP-complete to decide whether a given digraph has a decomposition into two strong spanning subdigraphs. This follows from the following result by Yeo (unpublished manuscript).

Theorem 5.1. *It is an NP-complete problem to decide whether a 2-regular digraph has two arc-disjoint hamiltonian cycles.*

Corollary 5.2. *It is NP-complete to decide whether a digraph contains two arc-disjoint spanning subdigraphs.*

We still believe that even for general digraphs there is a sufficient condition in terms of arc-connectivity for the existence of two arc-disjoint spanning strong subdigraphs.

Conjecture 12 (*Bang-Jensen and Yeo [19]*). *There exists a constant K so that every K-arc-strong digraph contains two arc-disjoint spanning strong digraphs.*

Note that every digraph with minimum degree k can be split into k parts so as to satisfy minimum degree at least one in each part:

Theorem 5.3. *One can always partition the arcs of a digraph D with $\delta(D) \geq k$ into k arc-disjoint spanning subdigraphs $H_1, H_2, \ldots, H_k$ such that $\delta(H_i) \geq 1$.*

Proof. It is enough to show that the bipartite representation $BG(D)$ of D (see [8, Page 25–26]) has a decomposition into k spanning bipartite graphs with minimum degree one. This follows from the classical result of Gupta [38] that a bipartite graph with minimum degree r has r edge-disjoint edge-covers. $\diamond$

5.2. Decompositions into vertex-disjoint subdigraphs

Another way of decomposing a digraph is to split it into several vertex-disjoint subdigraphs.

Problem 7 (*Thomassen [49]*). Does there exist a function $f(r, s)$ such that every $f(r, s)$-strong tournament T contains an r-strong tournament T_1 and an s-strong tournament T_2 such that $V(T) = V(T_1) \cup V(T_2)$ and $V(T_1) \cap V(T_2) = \emptyset$?

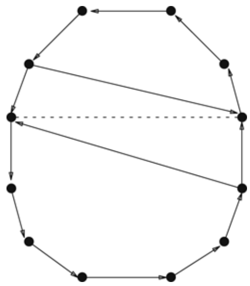

Fig. 5. A hamiltonian cycle in an extended semicomplete digraph which is not semicomplete will always be splittable into two complementary cycles as shown.

Two cycles in a digraph D are said to be *complementary* if they are disjoint and $V(C) \cup V(C') = V(D)$. For tournaments this corresponds to the case $r = s = 1$ above.

It follows from results by Reid and Song [49,50] that every 2-strong tournament on $n \geq 8$ vertices contains complementary cycles C_1, C_2 such that $|V(C_1)| = k$ and $|V(C_2)| = n - k$ for every $k = 3, 4, \ldots, n - 3$.

Answering a question of Bang-Jensen, Guo and Volkmann [37] proved that if D is a 2-strong locally semicomplete digraph which is not the second power of an odd cycle and has at least 9 vertices, then D has a pair of complementary cycles.

Problem 8. Is there a polynomial algorithm for deciding whether a semicomplete multipartite digraph has a pair of complementary cycles?

For the subclass of extended semicomplete digraphs there is an easy characterization which leads to a polynomial algorithm.

Theorem 5.4. *A strong extended semicomplete digraph D which is not semicomplete has complementary cycles if and only if it has a cycle factor and at most two strong components.*

Proof. Clearly both conditions are necessary. To prove sufficiency, first observe that a hamiltonian cycle can always be split into two complementary cycles when D is extended semicomplete and has a pair of non-adjacent vertices (see Fig. 5). Now the claim follows easily from the result by Gutin [39] that an extended semicomplete digraph is hamiltonian if and only if it is strong and has a cycle factor. $\diamond$

Conjecture 13 (*Volkmann [56]*). *With at most finitely many exceptions, every $(\alpha(D) + 1)$-strong semicomplete multipartite digraph D contains a pair of complementary cycles.*

Conjecture 13 is true for those semicomplete multipartite digraphs which are also extended semicomplete digraphs. This follows from the result above and the fact that every digraph D which is $\alpha(D)$-strong has a cycle factor (see [8, Proposition 3.11.6(c)]).

6. Cycle factors

The following problem is NP-hard for general digraphs, trivial for locally semicomplete and easy for extended semicomplete digraphs (by the result of Gutin above), but seems non-trivial for quasi-transitive digraphs and semicomplete multipartite digraphs. For quasi-transitive digraphs the problem was investigated in [16] where it was shown how to check for a cycle factor with at most 3 cycles in polynomial time. The problem is also open for semicomplete multipartite digraphs.

Problem 9 (*Bang-Jensen and Nielsen [16]*). Given a quasi-transitive digraph; find a cycle factor with the smallest possible number of cycles.

Problem 10. Let D be a digraph with complementary cycles. What is the smallest difference in D that one can achieve in the lengths of complementary cycles.

For 2-strong tournaments the answer is zero when n is even and one otherwise, by Song's theorem (see Section 5.2).

Problem 11. Characterize tournaments that have complementary cycles C, C' of the same length.

If T is 2-strong and n is even then such cycles always exist by Song's theorem. If $T - x$ is not strong with strong components $T_1, T_2, \ldots, T_k$ and for some $j \in \{2, \ldots, k - 1\}$ we have $|T_j| \geq n/2$ then the cycles exist (take an $n/2$-cycle C in T_j and a hamiltonian cycle in $T - V(C)$). So we may assume that $|T_1| \geq n/2$ (if no single T_i is large enough then the cycles cannot exist). Now the question involves solving the following problem, where X are the neighbours in T_i of x and $k = |T_1| - n/2$:

Problem 12. Given a strong tournament $T = (V, A)$ on n vertices a set $X \subset V$ and a number k: When does T have a path P on k vertices and a cycle C on $n - k$ vertices so that P and C are disjoint and P starts in a vertex from X?

Problem 13. Is there a polynomial algorithm which given a tournament T, decides whether T has a cycle factor $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ such that there exists $I \subset \{1, 2, \ldots, k\}$ with the property that

$$\sum_{i \in I} |C_i| = n/2?.$$

This problem is NP-complete for general digraphs as the partition problem easily reduces to it: Given an instance $\{x_1, x_2, \ldots, x_k\}$ of the partition problem let D consist of k disjoint cycles of lengths $x_1, x_2, \ldots, x_k$ respectively. What can be said about the complexity of the problem when restricted to various classes of tournament-like digraphs?

A related Conjecture is the following.

Conjecture 14 (*Song [50]*). *For any set of k integers $n_1, n_2, \ldots, n_k$, where each $n_i \geq 3$ and $\sum_{i=1}^{k} n_i = n$, all but a finite number of k-strong tournaments on n vertices contain a cycle factor with exactly k cycles such that the ith cycle has length n_i.*

Results related to this conjecture were obtained in [23,35].

7. Arc-disjoint hamiltonian paths and cycles

Although the structure of tournaments is very well studied, very little is known about digraphs which can be obtained from a tournament by deleting a small number of arcs. Such problems become relevant when we want to study the existence of arc-disjoint copies of the same subdigraph such as hamiltonian paths or cycles and also when we want to find a subdigraph avoiding certain arcs.

Problem 14 (*Bang-Jensen, Huang and Yeo [13]*). Which tournaments T contain a hamiltonian cycle C such that $\lambda(T - C) \geq \lambda(T) - 1$?

It follows from the family of tournaments in Fig. 3 that not all 2-strong tournaments satisfy this (every hamiltonian cycle will use all arcs going from right to left).

Conjecture 15 (*Bang-Jensen and Yeo [19]*). *Let T be an arbitrary tournament. Then either T contains two arc-disjoint hamiltonian cycles or T contains two arcs $a, a' \in A(T)$ such that $T - \{a, a'\}$ has no hamiltonian cycle.*

Conjecture 16 (*Thomassen [52]*). *Every 3-strong tournament contains two arc-disjoint hamiltonian cycles.*

By a result of Fraisse and Thomassen [30], every k-strong tournament contains a hamiltonian cycle avoiding any prescribed set of $k - 1$ arcs. Hence, if true, Conjecture 15 would imply Conjecture 16.

In Fig. 3 we can destroy all hamiltonian cycles by removing two arcs, but removing one is not enough.

Conjecture 17. *There exists a polynomial algorithm for deciding whether a given tournament contains two arc-disjoint hamiltonian cycles.*

A tournament is *almost transitive* if it can be obtained from a transitive tournament by reversing the arc from the vertex of maximum out-degree to the vertex of maximum in-degree. Thomassen [52] proved that a tournament T contains two arc-disjoint hamiltonian paths unless it has a strong component which is an almost transitive tournament of odd order or has two consecutive strong components of size 1, see Fig. 7.

Problem 15. Characterize those tournaments which contain two arc-disjoint hamiltonian paths with prescribed start vertices.

By inspection of Fig. 6 we see that no arc-strong connectivity suffices.

The following is a possible extension of the notion of hamiltonian connectivity. Note that $f(1) = 4$ by the result of Thomassen [51] that every 4-strong tournament is hamiltonian connected and examples in the same paper of 3-strong tournaments with no (x, y)-hamiltonian path.

Problem 16. Does there exist a function $f(k)$ such that every $f(k)$-strong tournament contains k arc-disjoint hamiltonian paths from x to y for every choice of x and y?

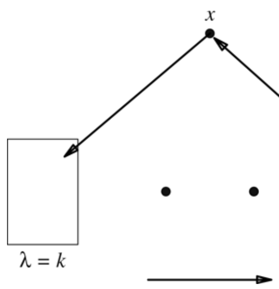

Fig. 6. An example showing that the arc-strong connectivity of a tournament T may be arbitrarily high and still there may be no two arc-disjoint hamiltonian cycles in T. The two boxes represent k-arc-strong tournaments and the two black dots form singleton strong components in $T - x$. In $T - x$ all arcs between components go from left to right and every vertex in the rightmost (leftmost) box dominates (is dominated by) x. It is easy to see that T is k-arc-strong and that every hamiltonian cycle uses the arc between the two middle components of $T - x$.

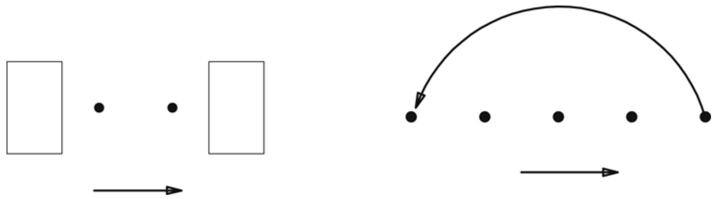

Fig. 7. Examples of tournaments with no two arc-disjoint hamiltonian paths. The boxes indicate arbitrary strong components and the black dots are singleton strong components in the left figure resp. single vertices in the right figure. All arcs not shown between components/vertices go from left to right.

8. Oriented hamiltonian paths

Havet and Thomassé [41] settled the longstanding conjecture of Rosenfeld that all but finitely many tournaments contain every orientation of a hamiltonian path. In fact the only exceptions are three tournaments on 3, 5 and 7 vertices that do not contain an anti-directed hamiltonian path (for short: an ADH-path) (A path is anti-directed if the orientations alternate between forward and backward e.g. $x_1 \rightarrow x_2 \leftarrow x_3 \rightarrow x_4 \leftarrow x_5 \ldots$). Havet [40] gave a polynomial algorithm which given any oriented path P on $n \geq 8$ vertices and a tournament T on n vertices will find an occurrence of P in T in polynomial time. Hell and Rosenfeld [42] gave a polynomial algorithm for testing the existence of an anti-directed hamiltonian path with prescribed end vertices in a tournament. Note that it is not specified that the first arc must be forward.

Problem 17 (*Hell and Rosenfeld [42]*). Extend the method of [42] to obtain a polynomial algorithm for deciding whether a given tournament T with vertices x, y has a forward ADH-path starting in x and ending in y.

If the last vertex is not specified, then the problem is polynomially solvable [4].

A block in an oriented path is a maximal sequence of forward or backward arcs.

Problem 18 (*Hell and Rosenfeld [42]*). Is it true that for any oriented path P on n vertices starting with a forward arc such that no block of P is larger than k, any tournament T on n vertices and any prescribed vertex x of T which has out-degree at least $k + 1$, there is an occurrence of P in T which starts in x on a forward arc?

As mentioned above, Thomassen [51] proved that there are 3-strong tournaments with no (x, y)-hamiltonian path for some choice of vertices x and y, but that every 4-strong tournament contains such a path. As pointed out in [42] a much weaker condition suffices to guarantee that a tournament T contains an anti-directed hamiltonian path with prescribed end vertices, namely it suffices that T has minimum in- and out-degree at least 4.

Problem 19 (*Hell and Rosenfeld*). Can the minimum degree bound of 4 above be lowered to 3 or even 2?

Problem 20. Find a sufficient condition for a tournament to contain two arc-disjoint ADH-paths with the same end vertices.

9. Cycles containing or avoiding prescribed arcs

An arc $x \rightarrow y$ in a digraph D is an *ordinary arc* if D does not contain the arc $y \rightarrow x$.

Conjecture 18 (*Tuza [54]*). *Let s be a positive integer and suppose that $D = (V, A)$ is a semicomplete digraph such that for every $X \subset V$ with $|X| < s$ the semicomplete digraph induced by $V - X$ is strong and has at least one ordinary arc. Then D contains a hamiltonian cycle with at least s ordinary arcs.*

Tuza has shown that the conjecture is true for $s = 1, 2$. He also proved the following.

Proposition 9.1 (*[54]*). *If a strong semicomplete digraph D has a cycle of length at least $s+1$ which contains at least s ordinary arcs, then D contains a hamiltonian cycle which contains at least s ordinary arcs.*

As mentioned earlier Fraisse and Thomassen [30] proved that every k-strong tournament has a hamiltonian cycle avoiding any prescribed set of $k-1$ arcs. This is a special case of the following difficult question. For a partial result which generalizes that in [30] see [10].

Problem 21 (*Bang-Jensen, Gutin and Yeo [10]*). Which sets B of edges of the complete graph K_n have the property that every k-strong tournament on n vertices induces a hamiltonian digraph on $K_n - B$?

Even special cases could be interesting to solve:

Problem 22 (*Bang-Jensen, Gutin and Yeo [10]*). What are sharp bounds for k above when B is a spanning forest of K_n consisting of m disjoint paths on $r_1, r_2, \dots, r_m$ vertices respectively? The same question can be asked for disjoint cycles or stars.

Conjecture 19 (*Bang-Jensen and Gutin [8]*). *For every k there exists a polynomial algorithm which given a semicomplete digraph $D = (V, A)$ and a subset $A' \subseteq A$, $|A'| = k$ decides whether D has a hamiltonian cycle that avoids all arcs in A'.*

Somewhat surprisingly this conjecture would follow from the following conjecture (see the details in [8, Section 6.7]).

Conjecture 20 (*Bang-Jensen, Manoussakis and Thomassen [15]*). *For every fixed k there is a polynomial algorithm for determining whether a given semicomplete digraph has a hamiltonian cycle containing k prescribed arcs.*

When $k = 1$, this follows from the result of [15] and when k is part of the input the problem is NP-complete [17].

10. Reversing arcs to increase connectivity

Problem 23. Is there a polynomial algorithm which given natural number k and a digraph $D = (V, A)$ on at least $k+1$ vertices, finds a minimum set $F \subset A$ of arcs in D such that the digraph D' obtained from D by reversing every arc in F is k-strong or determines that D has no such reversal?

If such a subset exists, then we let $r_k(D) = |F|$, where F is a minimum cardinality subset of A, whose reversal makes the resulting digraph k-strong. Otherwise we let $r_k(D) = \infty$. Determining whether $r_k(D)$ is finite seems to be a very hard problem for general digraphs (see the next section for orientation problems related to this fact).

Let $a_k(D)$ denote the minimum number of arcs we need to add to D to obtain a k-strong digraph. This number can be calculated in polynomial time for any digraph [33].

Clearly $a_k(D) \le r_k(D)$ always. Hence it is interesting to study classes for which the two numbers are equal.

Theorem 10.1 (*[14]*). *For every $k \ge 1$ and every semicomplete digraph D on at least $\max\{3, 3k-1\}$ vertices, $a_k(D) = r_k(D)$.*

Conjecture 21 (*Bang-Jensen and Jordan [14]*). *For every $k \ge 1$ and every tournament T on at least $2k+1$ vertices, $a_k(T) = r_k(T)$.*

For $k = 1$ this is trivial since every tournament has a hamiltonian path. Thus either $r_1(T) = 0$ if T is already strong or $r_1(T) = 1 = a_1(T)$.

The author made the following conjecture at a workshop in Budapest in 1994.

Conjecture 22 (*Bang-Jensen*). *For every natural number k and every tournament T on at least $2k+1$ vertices we have $r_k(T) \le k(k+1)/2$.*

It is not difficult to show that $r_k(TT_p) = k(k+1)/2$, for every $p \ge 2k+1$, where TT_p is the transitive tournament on p vertices. Hence the conjectured bound would be the best possible.

Let $m(k, D)$ denote the minimum number of arcs one needs to reverse in D in order to obtain a digraph D' with $\delta(D') \ge k$. If no such reversal exists then $m(k, D) = \infty$. The number $m(k, D)$ can be calculated and an optimal reversing set can be found in polynomial time using flows (see [20]).

Let $r'_k(D)$ denote the minimum number of arcs one needs to reverse in D in order to obtain a digraph D' which is k-arc-strong. By Nash-Williams' orientation theorem such a reversal exists if and only if the undirected multigraph we obtain from D by removing all orientations on the arcs (and keeping multiple edges) is $2k$-edge-connected (see [47] or [8, Page 443]). If no such reversal exists then $r'_k(D) = \infty$. The number $r'_k(D)$ and an optimal reversing set can be found in polynomial time using submodular flows even in the case when there are different costs on the arcs and the goal is to minimize the cost of the reversal (see [31] or [8, page 458]).

For tournaments the numbers $m(k, D)$ and $r'_k(D)$ are closely related. Note that we also have $r'_k(TT_p) = k(k+1)/2$, so that the bound below is the best possible.

Theorem 10.2 (*[20]*). *For every natural number k and every tournament T on $n \geq 2k+1$ vertices we have*

$$r'_k(T) = \max\{k - \lambda(T), m(k, D)\} \leq k(k+1)/2,$$

where $\lambda(T)$ denotes the arc-strong connectivity of T.

It follows from a result of Frank and Jordán [33] that $a_k(T) \leq k(k+1)$ for every tournament on at least $2k+1$ vertices. Here we give a simple proof for a weaker bound for r_k.

Theorem 10.3. *Every semicomplete digraph on $n \geq 2k+1$ vertices can be made k-strong by reversing the orientation of at most $\frac{(4k-2)(4k-3)}{4}$ arcs.*

Proof. First note the following two facts:

(a) If D is a k-strong digraph and D' is obtained from D by adding a new vertex x and arcs from x to every vertex in a set X of k distinct vertices of D and arcs from every vertex of a set Y of k distinct vertices of D to x, then D' is also k-strong.
(b) If T is a semicomplete digraph on at least $4k-1$ vertices, then T contains a vertex with in-degree and out-degree at least k.

By observations (a) and (b), for every semicomplete digraph T, $r_k(T) \leq r_k(T')$ for some subgraph T' of T with $|V(T')| \leq 4k-2$: Continue removing vertices as long as we can find a vertex of in- and out-degree at least k, or the current graph has $2k+1$ vertices. When this process stops we have $2k+1 \leq |V(T')| \leq 4k-2$ in the current semicomplete digraph T'. Then we can make T' k-strong by reversing some arcs and adding back each of the removed vertices in the reverse order of the deletion. This provides a simple upper bound for $r_k(T)$ (and hence for $a_k(T)$) as a function of k: we need to reverse at most half of the arcs, that is, at most $\frac{(4k-2)(4k-3)}{4}$ arcs. This follows from the fact that the k'th power of a hamiltonian cycle is k-strong. $\diamond$

11. Orientations which preserve vertex connectivity

As we indicated in the beginning of the last section, it is a difficult problem to determine whether a (multi)graph can be oriented to a directed (multi)graph with a prescribed guarantee on the vertex strong connectivity. As every graph has an equivalent digraph which is obtained by replacing every edge by a directed 2-cycle we can transform these orientation problems into what we call orientation problems for digraphs. By *orienting* a digraph we mean the operation where we delete one arc from every directed 2-cycle.

Jordán proved recently that every 18-connected graph has a 2-strong orientation [44]. Note that Jordán's result does not imply that every 18-strong digraph has a 2-strong orientation. Even the existence of a function $f = f(k)$ such that every $f(k)$-strong digraph has a k-strong orientation is open already for $k = 2$.

Conjecture 23 (*Jackson and Thomassen [53]*). *Every $2k$-strong digraph has a k-strong orientation.*

For several classes of digraphs that are close to tournaments we can give such a function.

Theorem 11.1 (*[36]*). *Every $(3k-2)$-strong locally semicomplete digraph has a k-strong orientation.*

The same bound holds for quasi-transitive digraphs (see [8, Page 400]).

It is interesting to note that even for semicomplete digraphs $3k-2$ is the best bound known so far for $k > 2$. For $k = 2$ this can be improved to 3 as shown by the author and Jordán (unpublished result).

For locally semicomplete digraphs which are not semicomplete Huang showed that we can always delete one arc from every 2-cycle without decreasing the vertex connectivity.

Theorem 11.2 (*[43]*). *If D is a k-strong locally semicomplete digraph which is not semicomplete, then D has a k-strong orientation.*

Finally we mention a conjecture for a necessary and sufficient condition for a graph to have a k-strong orientation.

Conjecture 24 (*Frank [32]*). *A graph G has a k-strong orientation if and only if for every $X \subset V(G)$ such that $|X| = j \leq k$ the graph $G - X$ is $2(k-j)$-edge-connected.*

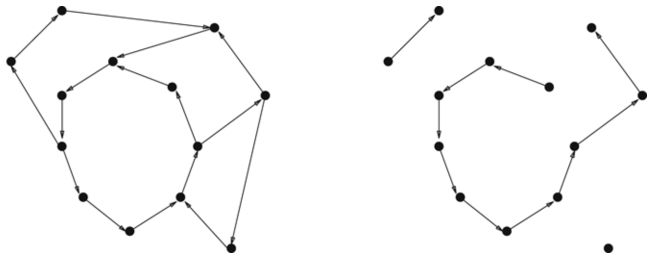

Fig. 8. Left: A digraph on n vertices and $n+3$ arcs. Right: A path covering with 3 paths.

12. Small certificates for strong connectivity

The MSSS problem is the following: Given a strong digraph $D = (V, A)$, find a spanning strong subdigraph $D' = (V, A')$ of D such that $|A'|$ is minimum.

The MSSS problem is clearly $\mathcal{NP}$-hard as it contains the hamiltonian cycle problem as a special case. Hence the problem can only be hoped to be polynomially solvable for those classes of digraphs where we know that the hamiltonian cycle problem is polynomial.

The path covering number of D, denoted by $pc(D)$, is the minimum number of vertex-disjoint paths needed to cover all vertices in D. Let $pc^*(D) = 0$ if D is hamiltonian and $pc^*(D) = pc(D)$ otherwise.

Lemma 12.1. *For every strongly connected digraph D, every spanning strong subdigraph of D has at least $n + pc^*(D)$ arcs. See Fig. 8*

Theorem 12.2 (*[18,12]*). *The MSSS problem is solvable in polynomial time for digraphs which are either quasi-transitive, semicomplete bipartite or extended semicomplete. Furthermore, if D is a strong digraph on n vertices from one of these classes, then the number of arcs in a minimum spanning strong subdigraph of D is exactly $n + pc^*(D)$, that is, it equals the lower bound of Lemma 12.1*

Conjecture 25 (*Bang-Jensen and Yeo [18]*). *The MSSS problem is polynomially solvable for semicomplete multipartite digraphs.*

It was shown in [18] that if D is a strong digraph which is either extended semicomplete or semicomplete bipartite on n vertices, then there exists a minimum spanning strong subdigraph D' of D which contains a longest cycle of D.

Problem 24. Does every strong semicomplete multipartite digraph D contain a minimum spanning strong subdigraph D' of D which contains a longest cycle of D?

Recall that it is an open problem whether we can find a longest cycle in a semicomplete multipartite digraph in polynomial time.

Theorem 12.3 (*[55]*). *There exists a polynomial algorithm which given a strong digraph $D = (V, A)$ returns a spanning strong subdigraph D' of D such that the number of arcs in D' is at most 1.5 times the number of arcs in a minimum spanning strong subdigraph of D.*

For a digraph D we denote by $\bar{\Delta}(D)$ the maximum degree of the complement of $UG(D)$, that is, the maximum number of non-neighbours of a vertex in D. If this number is bounded we can get a better bound when no vertex has more than $n/2$ non-neighbours.

Theorem 12.4 (*[13]*). *Every strong digraph D with $\bar{\Delta}(D) \le \frac{n}{r}$ contains a spanning strong digraph with at most $(1 + \frac{1}{r})n$ arcs. Furthermore, such a subdigraph can be found in polynomial time.*

In the weighted case, where each arc has a non-negative weight and the goal is to find a spanning strong subdigraph of minimum weight, the problem becomes NP-hard already for tournaments since it contains the hamiltonian cycle problem as a special case.

For arbitrary digraphs we can always get within a factor of two of the optimum by taking the union of a minimum weight out-branching and a minimum weight in-branching both rooted at the same vertex v [34]. Such branchings can be found in polynomial time (see [27] or [8, Section 9.10]).

Problem 25 (*Khuller, Raghavachari and Young [45]*). Does there exist an approximation algorithm for the weighted MSSS problem with a better approximation guarantee than 2?

Problem 26. Is there a polynomial approximation scheme for the weighted version of MSSS in the case of semicomplete digraphs?

13. Certificates for higher connectivities

Whereas the MSSS problem is trivial for tournaments (the answer is always a hamiltonian cycle) the corresponding problem MSSS_k ($\mathrm{MSSS}_k^{\mathrm{arc}}$) where we are looking for a minimum spanning k-(arc)-strong subdigraph of a k-(arc)-strong digraph is open even for the case of tournaments when $k \geq 2$.

By a *certificate* for the k-(arc)-strong connectivity of a digraph D we mean a spanning k-(arc)-strong subdigraph H of D. Clearly D itself is such a certificate so what we seek is a subdigraph H with as few arcs as possible. An optimal certificate for the k-(arc)-strong connectivity of D is a certificate H with the minimum number of arcs among all certificates for the k-(arc)-strong connectivity of D.

Problem 27. Determine the complexity of the problems MSSS_2 and $\mathrm{MSSS}_2^{\mathrm{arc}}$ for tournaments,

Approximation algorithms exist for MSSS_k and $\mathrm{MSSS}_k^{\mathrm{arc}}$ for general digraphs.

Theorem 13.1 (*[24]*). *There exists a polynomial algorithm which, given a digraph $D = (V, A)$ which is k-strong, returns a spanning k-strong subgraph $D'' = (V, A'')$ of D such that $|A''| \leq (1 + \frac{1}{k})|A^*_{opt}|$, where A^*_{opt} denotes a minimum cardinality arc set $A^*_{opt} \subseteq A$ such that $D^* = (V, A^*_{opt})$ is k-strong.*

Theorem 13.2 (*[24]*). *There exists a polynomial algorithm which given a digraph $D = (V, A)$ which is k-arc-strong returns a spanning k-arc-strong subgraph $D' = (V, A')$ of D such that $|A'| \leq (1 + 4/\sqrt{k})|A_{opt}|$, where $|A_{opt}|$ denotes the number of arcs in an optimal certificate for the k-arc-strong connectivity of D.*

A simple upper bound for the number of edges in the solution to $\mathrm{MSSS}_k^{\mathrm{arc}}$ can be obtained using branchings.

Proposition 13.3 (*[26]*). *Every k-arc-strong digraph on n vertices contains a spanning k-arc-strong digraph with at most $2k(n-1)$ arcs.*

Proof. Fix a vertex v and let $F^+_{v_1}, \ldots, F^+_{v_k}$ be arc-disjoint out-branchings rooted at v and $F^-_{v_1}, \ldots, F^-_{v_k}$ be arc-disjoint in-branchings rooted at v (existence follows from Edmonds' branching theorem). Let D' be the spanning subdigraph formed by taking the union of these $2k$ branchings. Then D' is k-arc-strong and has at most $2k(n-1)$ arcs. $\diamond$

For tournaments one can do a lot better. In fact we can get within an additive constant depending only on k of the obvious lower bound of nk arcs.

Theorem 13.4 (*[13]*). *For any $n \geq 3$ and $k \geq 1$, every k-arc-strong tournament T on n vertices contains a spanning k-arc-strong subdigraph D' with at most $nk + 136k^2$ arcs. Furthermore, such a spanning subdigraph can be found in polynomial time.*

For any tournament T we denote by $h(k, T)$ the minimum number of arcs in a spanning subdigraph D of T which has $\delta(D) \geq k$. If $\delta(T) < k$ we let $h(k, T) = \infty$.

The following result can be shown using flows (see [13]).

Proposition 13.5 (*[13]*). *For every tournament with $\delta(T) \geq k$ we have $h(k, T) \leq nk + k(k+1)/2$ and this is sharp. Furthermore, if T is k-arc-strong $h(k, T) \leq nk + k(k-1)/2$.*

For any tournament T we denote by $i(k, T)$ the minimum number of arcs in a spanning k-arc-strong subdigraph D of T. If T is not k-arc-strong then $i(k, T) = \infty$.

The next conjecture suggests a similar connection between $i(k, T)$ and $h(k, T)$ as the one described for reversals in Theorem 10.2.

Conjecture 26 (*Bang-Jensen, Huang and Yeo [13]*). *For every natural number k and every k-arc-strong tournament T we have $i(k, T) = h(k, T)$.*

We finish with the following question related to MSSS_k for tournaments.

Problem 28. Does there exist a function $g = g(k)$ such that every k-strong tournament contains a spanning k-strong subdigraph with at most $kn + g(k)$ arcs?

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References

[1] N. Ailon, N. Alon, Hardness of fully dense problems, Inform. and Comput. 205 (8) (2007) 1117–1129.
[2] N. Alion, M. Charikar, A. Newman, Aggregating inconsistent information: Ranking and clustering, in: Proceedings of the 37th Symposium on the Theory of Computing, STOC, ACM Press, 2005, pp. 684–693.

[3] N. Alon, Ranking tournaments, SIAM J. Discrete Math. 20 (2006) 137–142.
[4] E. Bampis, P. Hell, Y. Manoussakis, M. Rosenfeld, Finding an antidirected hamiltonian path starting with a forward arc from a given vertex in a tournament, Lect. Notes Comput. Sci. 1120 (1996) 67–73.
[5] J. Bang-Jensen, Edge-disjoint in- and out-branchings in tournaments and related path problems, J. Combin. Theory Ser. B 51 (1) (1991) 1–23.
[6] J. Bang-Jensen, Digraphs with the path-merging property, J. Graph Theory 20 (2) (1995) 255–265.
[7] J. Bang-Jensen, G. Gutin, Generalizations of tournaments: A survey, J. Graph Theory 28 (1998) 171–202.
[8] J. Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, Springer-Verlag, London, 2001.
[9] J. Bang-Jensen, G. Gutin, J. Huang, Weakly Hamiltonian-connected ordinary multipartite tournaments, in: 14th British Combinatorial Conference (Keele, 1993), Discrete Math. 138 (1–3) (1995) 63–74.
[10] J. Bang-Jensen, G. Gutin, A. Yeo, Hamiltonian cycles avoiding prescribed arcs in tournaments, Combin. Probab. Comput. 6 (3) (1997) 255–261.
[11] J. Bang-Jensen, G. Gutin, A. Yeo, A polynomial algorithm for the Hamiltonian cycle problem in semicomplete multipartite digraphs, J. Graph Theory 29 (1998) 111–132.
[12] J. Bang-Jensen, J. Huang, A. Yeo, Strongly connected spanning subdigraphs with the minimum number of arcs in quasi-transitive digraphs, SIAM J. Discrete Math. 16 (2003) 335–343.
[13] J. Bang-Jensen, J. Huang, A. Yeo, Spanning k-arc-strong subdigraphs with few arcs in k-arc-strong tournaments, J. Graph Theory 46 (2004) 265–284.
[14] J. Bang-Jensen, T. Jordán, Adding and reversing arcs in semicomplete digraphs, Combin. Probab. Comput. 7 (1) (1998) 17–25.
[15] J. Bang-Jensen, Y. Manoussakis, C. Thomassen, A polynomial algorithm for Hamiltonian-connectedness in semicomplete digraphs, J. Algorithms 13 (1) (1992) 114–127.
[16] J. Bang-Jensen, M.H. Nielsen, Minimum cycle factors in quasi-transitive digraphs, Technical Report 7, Department of Mathematics and Computer Science, 2004.
[17] J. Bang-Jensen, C. Thomassen, A polynomial algorithm for the 2-path problem for semicomplete digraphs, SIAM J. Discrete Math. 5 (1992) 366–376.
[18] J. Bang-Jensen, A. Yeo, The minimum spanning strong subdigraph problem for extended semicomplete digraphs and semicomplete bipartite digraphs, J. Algorithms 41 (2001) 1–19.
[19] J. Bang-Jensen, A. Yeo, Decomposing k-arc-strong tournaments into strong spanning subdigraphs, Combinatorica 24 (2004) 331–349.
[20] J. Bang-Jensen, A. Yeo, Making a tournament k-arc-strong by reversing or deorienting arcs, Discrete Appl. Math. 136 (2004) 161–171.
[21] B. Bollobás, R. Häggkvist, Powers of Hamilton cycles in tournaments, J. Combin. Theory Ser. B 50 (2) (1990) 309–318.
[22] P. Charbit, P. Thomassé, A. Yeo, The minimum Feedback arc set problem is NP-hard for tournaments, Combin. Probab. Comput. 16 (1) (2007) 1–4.
[23] G. Chen, R. Gould, H. Li, Partitioning the vertices of a tournament into independent cycles, J. Combin. Theory, Ser. B 83 (2001) 213–220.
[24] J. Cheriyan, R. Thurimella, Approximating minimum-size k-connected spanning subgraphs via matching, Siam J. Comput. 30 (2000) 528–560.
[25] V. Conitzer, Computing Slater rankings using similarities among candidates, in: Proceedings, The Twenty-First National Conference on Artificial Intelligence and the Eighteenth Innovative Applications of Artificial Intelligence Conference, July 16–20, AAAI Press, Boston, Massachusetts, USA, 2006.
[26] M. Dalmazzo, Nombre d'arcs dans les graphes k-arc-fortement connexes minimaux, C.R.Acad. Sci. Paris A 285 (1977) 341–344.
[27] J. Edmonds, Optimum branchings, J. Res. Natl. Bur. Stand. Sect. B 71B (1967) 233–240.
[28] J. Edmonds, Edge-disjoint branchings, in: B. Rustin (Ed.), Combinatorial Algorithms, Academic Press, New York, 1973, pp. 91–96.
[29] A. El Sahili, Trees in tournaments, J. Combin. Theory, Ser. B 92 (2004) 183–187.
[30] P. Fraisse, C. Thomassen, Hamiltonian dicycles avoiding prescribed arcs in tournaments, Graphs Combin. 3 (3) (1987) 239–250.
[31] A. Frank, An algorithm for submodular functions on graphs, in: Bonn Workshop on Combinatorial Optimization (Bonn, 1980), in: Ann. Discrete Math., vol. 16, North-Holland, Amsterdam, 1982, pp. 97–120.
[32] A. Frank, Connectivity and network flows, in: Handbook of combinatorics, vol. 1, 2, Elsevier, Amsterdam, 1995, pp. 111–177.
[33] A. Frank, T. Jordán, Minimal edge-coverings of pairs of sets, J. Combin. Theory Ser. B 65 (1) (1995) 73–110.
[34] G.N. Frederickson, J. Ja'Ja', Approximation algorithms for several graph augmentation problems, SIAM J. Comput. 10 (1981) 270–283.
[35] R. Gould, Y. Guo, Locally semicomplete digraphs with a factor composed of k cycles, J. Korean Math. Soc. 41 (2004) 895–912.
[36] Y. Guo, Spanning local tournaments in locally semicomplete digraphs, in: 4th Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1995), Discrete Appl. Math. 79 (1997) 119–125.
[37] Y. Guo, L. Volkmann, On complementary cycles in locally semicomplete digraphs, Discrete Math. 135 (1–3) (1994) 121–127.
[38] R.P. Gupta, The chromatic index and the degree of a graph (Abstract 66T-429), Notices Amer. Math. Soc. (1966).
[39] G. Gutin, Cycles and paths in directed graphs, Ph.D. Thesis, School of Mathematics, Tel Aviv University, 1993.
[40] F. Havet, Finding an oriented hamiltonian path in a tournament, J. Algorithms 36 (2000) 253–275.
[41] F. Havet, S. Thomassé, Oriented hamiltonian paths in tournaments: A proof of Rosenfeld's conjecture, J. Combin. Theory, Ser B 78 (2000) 243–273.
[42] P. Hell, M. Rosenfeld, Antidirected hamiltonian paths between specified vertices of a tournament, Discrete Appl. Math. 117 (2002) 87–98.
[43] J. Huang, A note on spanning local tournaments in locally semicomplete digraphs, Discrete Appl. Math. 89 (1998) 277–279.
[44] T. Jordán, On the existence of k edge-disjoint 2-connected spanning subgraphs, J. Combin. Theory, Ser. B 95 (2005) 257–262.
[45] S. Khuller, B. Raghavachari, N. Young, Approximating the minimum equivalent digraph, SIAM J. Comput. 24 (4) (1995) 859–872.
[46] J.W. Moon, Topics on Tournaments, Holt, Rinehart and Winston, New York, 1968.
[47] C.St.J.A. Nash-Williams, On orientations, connectivity and odd-vertex-pairings in finite graphs, Canad. J. Math. 12 (1960) 555–567.
[48] V. Raman, S. Saurabh, Parameterized complexity of directed feedback arc set problems in tournaments, in: Algorithms and Data Structures, in: Lecture Notes in Computer Science, vol. 2748, Springer, Berlin, 2003, pp. 484–492.
[49] K.B. Reid, Two complementary circuits in two-connected tournaments, in: Cycles in Graphs (Burnaby, B.C., 1982), in: North-Holland Math. Stud., vol. 115, North-Holland, Amsterdam, 1985, pp. 321–334.
[50] Z.M. Song, Complementary cycles of all lengths in tournaments, J. Combin. Theory Ser. B 57 (1) (1993) 18–25.
[51] C. Thomassen, Hamiltonian-connected tournaments, J. Combin. Theory Ser. B 28 (2) (1980) 142–163.
[52] C. Thomassen, Edge-disjoint Hamiltonian paths and cycles in tournaments, Proc. London Math. Soc. (3) 45 (1) (1982) 151–168.
[53] C. Thomassen, Configurations in graphs of large minimum degree, connectivity, or chromatic number, Ann. New York Acad. Sci. 555 (1989) 402–412.
[54] Z. Tuza, Characterization of $(m, 1)$-transitive and (3,2)-transitive semi-complete directed graphs, Discrete Math. 135 (1–3) (1994) 335–347.
[55] A. Vetta, Approximating the minimum strongly connected subgraph via a matching lower bound, in: Proceedings of the twelfth ACM-Siam Symposium on Discrete Algorithms, Washington, DC 2001, Siam, Philadelphia, 2001, pp. 417–426.
[56] L. Volkmann, Cycles in multipartite tournaments: Results and problems, Discrete Math. 245 (2002) 19–53.
[57] N.C. Wormald, Subtrees of large tournaments, Lecture Notes in Math. 1036 (1983) 417–419.
[58] A. Yeo, A polynomial time algorithm for finding a cycle covering a given set of vertices in a semicomplete multipartite digraph, J. Algorithms 33 (1) (1999) 124–139.